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The onset of transient convective instability

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The stability of a horizontal fluid layer when the thermal (or concentration) gradient is not uniform is examined by means of linear stability analysis. Both buoyancy and surface-tension effects are considered, and the analogous problem for a porous medium is also treated. Attention is focused on the situation where the critical Rayleigh number (or Marangoni number) is less than that for a linear thermal gradient, and the convection is not (in general) maintained. The case of constant-flux boundary conditions is examined because then a simple application of the Galerkin method gives useful results and general basic temperature profiles are readily treated. Numerical results are obtained for special cases, and some general conclusions about the destabilizing effects, with respect to disturbances of infinitely long wavelength, of various basic temperature profiles are presented. If the basic temperature gradient (considered positive, for a fluid which expands on heating, if the temperature decreases upwards) is nowhere negative, then the profile which leads to the smallest critical Rayleigh (or Marangoni) number is one in which the temperature changes stepwise (at the level at which the velocity, if motion were to occur, would be vertical) but is otherwise uniform. If, as well as being non-negative, the temperature gradient is a monotonic function of the depth, then the most unstable temperature profile is one for which the temperature gradient is a step function of the depth.

1. Introduction

The determination of the criterion for the onset of instability as convection in a horizontal layer of fluid heated uniformly from below is a classical problem associated with the names of Bénard and Rayleigh. The copious literature on this problem and its extensions has been reviewed many times, notably by Chandrasekhar (1961), Segel (1966), Berg, Acrivos & Boudart (1966), Brindley (1967), Spiegel (1971, 1972), Schechter & Velarde (1974) and Koschmieder (1974). The analogous problem in a saturated medium is also of interest (Nield 1968). In general, convection appears when a certain dimensionless parameter exceeds its critical value. This parameter is a Rayleigh number (thermal or solutal) when the convection is induced by buoyancy effects due to variations in density and is a Marangoni number when surface-tension variations induce the convection.

In the classical problem the basic temperature distribution is the steady-state (conduction) distribution, the temperature gradient being constant. However, in many situations (in particular, in geophysical contexts) the stability or instability of a fluid in the presence of a nonlinear (and usually time-dependent) temperature profile is of practical importance. An early study in this field was that of Malurkar (1937), who investigated the stability of a radiating layer of air near the ground. Often the nonlinearity of the temperature profile is due to rapid heating (or cooling) at a boundary. The experiments of Graham (1933) and Chandra (1938) attracted attention because they found that in thin layers a form of convection appeared at values of the overall Rayleigh number lower than the critical value predicted by the classical theory. Sutton (1950) noted that a likely explanation of this phenomenon was the non-uniformity of the temperature gradient in such layers.

Further theoretical studies of instability with a nonlinear basic temperature profile were made by Morton (1957), Goldstein (1959) and Lick (1965). In these studies, as in that by Foster (1965), the emphasis was on calculating the rates of growth of disturbances of various wavelengths. Foster noted that the assumption, made in the previous studies, that the temperature distribution could be taken as quasi-static led to inaccurate estimates of the growth rates, and showed how this difficulty could be avoided. The validity of the quasi-static assumption was further discussed by Robinson (1967) and by Gresho & Sani (1971). The quasi-static assumption was also made by Currie (1967), who showed that, with a piecewise linear profile, the onset of convection could occur at a Rayleigh number considerably smaller than the critical value for a linear profile. Analysis similar to that of Currie was performed by Ogura & Kondo (1970), by Takaki & Yoshizawa (1972) and by Kondo, Kimura & Nishimoto (1972). The effects of modulating the basic temperature profile with time have been considered by several authors, including Gershuni & Zhukhovitskii (1963), Venezian (1969), Rosenblat & Herbert (1970), Rosenblat & Tanaka (1971), Burde (1971) and Yih & Li (1972), and the effect of changing the mean temperature was discussed by Krishnamurti (1968). The onset of convection with a parabolic basic temperature profile was first treated by Sparrow, Goldstein & Jonsson (1964), and subsequently by Roberts (1967). Global stability of time-dependent profiles has been treated using the energy method by Homsy (1973).

In all the above papers the agency responsible for the convection was buoyancy. A problem similar to that treated by Currie (1967), but with convection driven by surface-tension gradients rather than by buoyancy forces, was analysed by Vidal & Acrivos (1968). They failed to note that convection could occur at a Marangoni number less than that predicted for a linear profile because they defined their Marangoni number in terms of the maximum temperature gradient instead of the mean one. When we calculate the overall Marangoni number from their data, we find that it goes through a minimum, just as the Rayleigh number does in figure 2 of Currie (1967). The surface-tension problem with a parabolic basic temperature profile, for which the critical Marangoni number is less than that for a linear profile, was treated by Debler & Wolf (1970).

Little work has been done on the onset of convection (induced by buoyancy) in a saturated porous medium with a nonlinear basic temperature distribution since the time of the *ad hoc* investigations by Rogers & Morrison (1950), Rogers, Schilberg & Morrison (1951) and Rogers (1953). However, a parabolic distribution was discussed by List (1965).

Once we accept the fact that a piecewise-linear basic temperature profile can lead to convection at an overall Rayleigh number less than the critical value for a completely linear profile, the question arises as to which temperature profile gives the absolute minimum. If we allow an unrestricted choice of profile the question is trivial, because Sparrow *et al.* (1964) showed that the critical overall Rayleigh number for a parabolic profile arising from a uniform volume distribution of heat sources could be arbitrarily small. However, if we restrict the profiles to ones in which the gradient does not change sign, the answer is not immediately obvious.

In the present paper we attempt to find the answer in this case. We examine the stability (with respect to small disturbances) of a layer in the presence of a 'frozen' basic temperature profile. Since we are not concerned with the magnitude of the growth rate of the disturbances, but only with whether or not they grow at all at any stage of the evolution of the basic temperature profile, the arguments of Foster and others against the use of 'frozen' profiles are not applicable. Even with this simplification, we are left, for general boundary conditions, with a formidable problem, because we not only need to minimize a Rayleigh or Marangoni number with respect to variation of the horizontal wavenumber of the disturbances, but also have to consider a whole class of basic temperature profiles. Fortunately, for the case of constant-flux (of heat or solute) conditions at both horizontal boundaries, the small wavenumber approximation is applicable, and as a consequence a simple Galerkin method turns out to give some useful results. (Convection does occur at small wavenumbers in situations such as that discussed by Sasaki 1970.)

From the work of Davis (1972) it appears that, when the overall Rayleigh number R is less than the critical value R_L for a linear profile, the disturbance must ultimately decay. Hence, if one does have instability with $R < R_L$, the convection will be transitory. For some purposes it may not matter whether or not such transient convection occurs, but in some industrial processes it is vitally important to ensure that convection does not occur at any stage. In fact, it is hoped that the virtual absence of buoyancy effects in a space vehicle will enable improved products to be made. However, if the fluid has a free surface then, as experiments on Apollo 14 and 17 flights (reported by Grodzka & Bannister 1972; and Bannister *et al.* 1973) have shown, convection can still be induced by surface-tension effects even if buoyancy forces are absent, and hence it is of importance to calculate the critical Marangoni number below which convection cannot occur.

In succeeding sections we discuss, in turn, buoyancy effects in a fluid layer, surface-tension effects in a fluid layer and buoyancy effects in a layer of a saturated porous medium. In § 5 we present some general conclusions.

2. Buoyancy effects in a fluid layer

The procedure for obtaining the linearized perturbation equations is now well known, so we omit the details (which are discussed fully by Chandrasekhar 1961, chap. 2). We find it convenient to follow the exposition of Finlayson (1972, §6.2). The fluid is assumed to be bounded by horizontal planes a distance d apart. At each boundary the basic temperature is assumed uniform, that of the lower boundary being ΔT greater than that of the upper boundary. Cartesian axes are taken with origin on the lower boundary and the z axis vertically upwards. For disturbances having horizontal wavenumber a, the vertical component of the velocity and W the perturbation temperature T (the total temperature minus the basic temperature T_0) are related by the following equations (which are derived from the Navier–Stokes and heat-conservation equations for a Boussinesq fluid):

$$\partial[(D^2-a^2)\,W]/\partial t = -R^{\frac{1}{2}}aT+(D^2-a^2)^2\,W, \tag{2.1}$$

$$Pr\,\partial T/\partial t = (D^2-a^2)\,T+R^{\frac{1}{2}}af(z)\,W, \tag{2.2}$$

where $D \equiv \partial/\partial z, f(z) = (-d/\Delta T)\,dT_0/dz$, $R = g\alpha d^3\Delta T/\nu\kappa$ is the Rayleigh number, $Pr = \nu/\kappa$ is the Prandtl number, g is the gravitational acceleration, α is the volume coefficient of thermal expansion, ν is the kinematic viscosity and κ is the thermal diffusivity. The time scale has been chosen to be d^2/ν and the length scale to be d, so that the lower and upper boundaries are $z = 0$ and $z = 1$ respectively. The non-dimensional temperature gradient $f(z)$ must satisfy

$$\int_0^1 f(z)\,dz = 1.$$

The scales for W and T have been chosen such that R appears symmetrically in the two equations, rather than in just one or the other. Though this choice is not essential for our present purpose, it enables a variational principle to be established for the present set of equations (and appropriate boundary conditions) and the adjoint set, and as Finlayson (1972, §6.4) shows, this leads to the conclusion that the eigenvalue R is stationary in the Galerkin method which we shall apply below.

As usual, we consider boundaries which are either rigid (so that $W = DW = 0$ on the boundary) or stress free (so that $W = D^2W = 0$ there). For the reason given above, we confine our attention to boundaries on which the basic heat flux is kept constant, so that $DT = 0$ on each boundary.

We now apply the Galerkin method as described by Finlayson (1972, §6.2). We shall find that we can get an accurate value for the critical value of R for certain cases by taking a single term in the expansions for W and T, i.e. we let $W = AW_1$ and $T = BT_1$, where W_1 and T_1 are suitably chosen trial functions. Further, we take advantage of the fact that oscillatory convection is ruled out for a constant-property fluid subject to a single agency capable of causing instability, because there is then no mechanism able to produce the oscillations. The neutral-stability solution is thus one for which the time derivatives in the differential equations (2.1) and (2.2) are zero. These equations then yield

$$R^{\frac{1}{2}}a\langle WT\rangle = -\langle (D^2W)^2+2a^2(DW)^2+a^4W^2\rangle,$$

$$R^{\frac{1}{2}}a\langle Wf(z)\,T\rangle = \langle (DT)^2+a^2T^2\rangle.$$

Here angular brackets denote integration across the layer. We have performed some integrations by parts and used the boundary conditions. Substituting $W = AW_1$ and $T = BT_1$, eliminating A and B and dropping the suffixes, we get

$$R = \frac{\langle (D^2W)^2 + 2a^2(DW)^2 + a^4W^2\rangle \langle (DT)^2 + a^2T^2\rangle}{a^2\langle WT\rangle \langle Wf(z)\,T\rangle}. \tag{2.3}$$

We now select trial functions satisfying the appropriate sets of boundary conditions. We consider two cases.

2.1. *Both boundaries rigid*

The boundary conditions are

$$W = DW = DT = 0 \quad \text{at} \quad z = 0, 1. \tag{2.4}$$

These are satisfied by

$$W = z^2(1-z)^2, \quad T = 1.$$

Equation (2.3) then yields

$$R = 24(1 + \tfrac{1}{21}a^2 + \tfrac{1}{504}a^4)/\langle (z^2 - 2z^3 + z^4) f(z)\rangle.$$

For any given $f(z)$, R attains its minimum when $a = 0$, and its minimum value is

$$R_c = 24/\langle (z^2 - 2z^3 + z^4) f(z)\rangle. \tag{2.5}$$

For a linear profile, $f(z) = 1$ and $R_c = 720$. This is in fact the known exact value for R_c.

As an example (chosen for reasons discussed in §5), for the piecewise-linear profile given by

$$f(z) = \begin{cases} \epsilon^{-1}, & 0 \leqslant z < \epsilon, \\ 0, & \epsilon < z \leqslant 1, \end{cases} \tag{2.6}$$

we have

$$R_c = 720/(6\epsilon^4 - 15\epsilon^3 + 10\epsilon^2). \tag{2.7}$$

As ϵ increases from 0 to 1, R_c decreases from $+\infty$ to a minimum value of

$$720/1{\cdot}198 = 601{\cdot}1 \quad \text{at} \quad \epsilon = 0{\cdot}724$$

and then increases to 720 at $\epsilon = 1$. A similar variation of R_c with ϵ was found by Currie (1967) for the case of constant-temperature rather than constant-flux boundaries. In his case the minimum was attained at $\epsilon = 0{\cdot}72$ and the minimum value was 1340, a reduction by a factor 1·274 from the well-known value of 1707·8, at $\epsilon = 1$, for a linear profile. Compared with the constant-flux condition, the constant-temperature condition is more restricting, so that in the latter case there is a greater potential for reduction of the eigenvalue when the buoyancy forces are applied in a restricted area, away from one of the boundaries.

One might expect that an even greater reduction might be possible if the buoyancy forces were applied in the middle of the layer, away from both boundaries. This is indeed so. For example, for the step-function profile in which the basic temperature drops suddenly by an amount ΔT at $z = \epsilon$ but is otherwise uniform we have

$$f(z) = \delta(z - \epsilon), \tag{2.8}$$

where δ denotes the Dirac delta function. Now

$$R_c = 720/(\epsilon^2 - 2\epsilon^3 + \epsilon^4), \tag{2.9}$$

which has a minimum value of $720 \times \frac{8}{15} = 384$, attained at $\epsilon = \frac{1}{2}$, i.e. midway between the boundaries.

2.2. *Lower boundary rigid, upper boundary free*

The boundary conditions are now

$$\left.\begin{aligned} W = DW = DT = 0 \quad \text{at} \quad z = 0, \\ W = D^2W = DT = 0 \quad \text{at} \quad z = 1. \end{aligned}\right\} \tag{2.10}$$

These are satisfied by

$$W = z^2(1-z)(3-2z), \quad T = 1.$$

Again R attains its minimum when $a = 0$, and

$$R_c = 48/\langle(3z^2 - 5z^2 + 2z^4) f(z)\rangle. \tag{2.11}$$

When $f(z) = 1$, $R_c = 320$. Again the approximate Galerkin method has given the known exact value of R_c for a linear profile.

For the piecewise-linear profile given by (2.6) (which approximates the profile for heating from below),

$$R_c = 960/(20\epsilon^2 - 25\epsilon^2 + 8\epsilon^4). \tag{2.12}$$

This has a minimum of $320/1{\cdot}094 = 292{\cdot}5$, attained at $\epsilon = 0{\cdot}821$ (further away from the rigid boundary than for rigid–rigid conditions). For the piecewise-linear profile given by

$$f(z) = \left\{\begin{matrix} 0, & 0 \leqslant z < 1-\epsilon, \\ \epsilon^{-1}, & 1-\epsilon < z \leqslant 1 \end{matrix}\right\} \tag{2.13}$$

(approximating the profile for cooling from above), we have

$$R_c = 960/(10\epsilon - 15\epsilon^3 + 8\epsilon^4). \tag{2.14}$$

This has a minimum value of $320/1{\cdot}270 = 252{\cdot}0$, attained when $\epsilon = 0{\cdot}638$ and thus $1-\epsilon = 0{\cdot}362$. Comparing the last two results, we see that R_c is less when the buoyancy force is applied near the less restrictive boundary (the free one) rather than the more restrictive boundary (the rigid one).

For the step-function profile with $f(z)$ given by (2.8) we have

$$R_c = 48/(3\epsilon^2 - 5\epsilon^3 + 2\epsilon^4),$$

which has a minimum value of $320/1{\cdot}733 = 184{\cdot}6$, attained at $\epsilon = 0{\cdot}578$. Thus, as we would expect, the most destabilizing step-function profile has the step closer to the free boundary than to the rigid one.

3. Surface-tension effects in a fluid layer

Again we closely follow the treatment by Finlayson (1972). In the absence of buoyancy forces $R = 0$, and in place of (2.1) and (2.2) we have

$$\partial[(D^2 - a^2) W]/\partial t = (D^2 - a^2)^2 W, \tag{3.1}$$

$$Pr\, \partial T/\partial t = (D^2 - a^2) T + M^{\frac{1}{2}} a f(z) W, \tag{3.2}$$

where $M = \sigma d\Delta T/\mu\kappa$ is the Marangoni number, μ is the dynamic viscosity and σ is the rate of decrease of surface tension with increasing temperature. The boundary conditions for a rigid bottom and a free upper surface with temperature-dependent surface tension, each subject to constant heat flux, are

$$\left.\begin{aligned} W = DW = DT = 0 \quad &\text{at} \quad z = 0,\\ W = D^2W + M^{\frac{1}{2}}aT = DT = 0 \quad &\text{at} \quad z = 1.\end{aligned}\right\} \tag{3.3}$$

The trial functions $W_1 = (1-z)z^2$ and $T_1 = 1$ satisfy all the boundary conditions except one, namely $D^2W + M^{\frac{1}{2}}aT = 0$ at $z = 1$, and a residual from this equation is included in a residual from the differential equations. For neutral stability we may again put $\partial/\partial t = 0$; then (3.1) and (3.2) yield

$$\langle (D^2W)^2 + 2a^2(DW)^2 + a^4W^2\rangle + M^{\frac{1}{2}}aDW(1)\,T(1) = 0,$$

$$\langle (DT)^2 + a^2T^2\rangle = M^{\frac{1}{2}}a\langle Wf(z)\,T\rangle.$$

When we substitute $W = AW_1$ and $T = BT_1$, eliminate A and B, and drop the suffixes, we have

$$M = \frac{-\langle (D^2W)^2 + 2a^2(DW)^2 + a^4W^2\rangle\langle (DT)^2 + a^2T^2\rangle}{a^2DW(1)\,T(1)\langle Wf(z)\,T\rangle},$$

and on substituting our trial functions we get

$$M = 4(1 + \tfrac{1}{15}a^2 + \tfrac{1}{420}a^4)/\langle (z^2 - z^3)f(z)\rangle.$$

Thus M is least when $a = 0$ and

$$M_c = 4/\langle (z^2 - z^3)f(z)\rangle. \tag{3.4}$$

For the linear profile, $f(z) = 1$ and $M_c = 48$, the known exact value for the critical Marangoni number for this case. For the bottom-heating piecewise-linear profile given by (2.6) we have

$$M_c = 48/(4\epsilon^2 - 3\epsilon^3). \tag{3.5}$$

This has a minimum of $48 \times \frac{243}{256} = 48/1{\cdot}0535 = 45\frac{9}{16}$, attained at $\epsilon = \frac{8}{9}$. For the top-cooling piecewise-linear profile given by (2.13) we have

$$M_c = 48/(3\epsilon^3 - 8\epsilon^2 + 6\epsilon). \tag{3.6}$$

This has a minimum value of $48/1{\cdot}380 = 34{\cdot}79$. As we expect, cooling from above is more effective than heating from below in causing instability in this case. Our approximate values of M_c given by (3.6) may be compared with the numerical data of Vidal & Acrivos (1968). We see from table 1 that (3.6) gives upper bounds on M_c which are good approximations for $\epsilon \geqslant 0{\cdot}4$ (a range which includes the value giving the minimum value of M_c) because then the exact critical wave-number differs little from zero.

Debler & Wolf (1970) have considered the problem with a parabolic distribution in which the basic temperature gradient is zero at the lower boundary, corresponding to $f(z) = 2z$ here. Equation (3.4) now gives $M_c = 40$, the value for the Marangoni number given by their equation (6). (The corresponding curve in their figure 1 is slightly inaccurate.) For comparison the 'inverted' parabolic

ϵ	M for $a = 0$	M_c	a_c
0	∞	∞	—
0·05	169·16	104·29	3·3
0·1	91·78	64·56	2·7
0·4	36·58	36·0	1·0
0·5	34·91	34·9	0·1
0·8	39·47	39·5	0·001
1·0	48·00	48·0	0

TABLE 1. Values of the critical Marangoni number M_c and the corresponding wavenumber a_c for various values of the thermal depth parameter ϵ when the lower boundary is rigid and the heat flux is constant at each boundary. Column 2 contains the approximate values of M_c given by (3.6), which are in fact the exact values of M when $a = 0$. Columns 3 and 4 contain the values calculated by Vidal & Acrivos (1968), with the Marangoni number defined as $\sigma d\Delta T/\mu\kappa$, that is, their Marangoni number multiplied by ϵ.

profile with $f(z) = 2(1-z)$ gives $M_c = 60$ and, as expected on physical grounds, is less destabilizing.

For the step-function profile with $f(z) = \delta(z-\epsilon)$, we have

$$M_c = 4/(\epsilon^2-\epsilon^3),$$

which has a minimum value of $48 \times \frac{9}{16} = 27$, attained at $\epsilon = \frac{2}{3}$, values again in accord with our expectation.

4. Buoyancy effects in a saturated porous medium

The appropriate time-independent equations are

$$(D^2-a^2)\,W = -R^{\frac{1}{2}}aT, \tag{4.1}$$

$$(D^2-a^2)\,T = -R^{\frac{1}{2}}af(z)\,W, \tag{4.2}$$

where R now denotes the quantity $g\alpha Kd\Delta T/\nu\kappa$, where K is the permeability of the medium.

For impermeable boundaries through which the heat flux is constant we have

$$W = DT = 0 \quad \text{at} \quad z = 0, 1. \tag{4.3}$$

With trial functions $W_1 = z(1-z)$ and $T_1 = 1$, which satisfy the boundary conditions, the Galerkin method yields

$$R = \frac{\langle (DW)^2+a^2W^2\rangle \langle (DT)^2+a^2T^2\rangle}{a^2\langle WT\rangle \langle Wf(z)\,T\rangle}, \tag{4.4}$$

giving

$$R = (2+\tfrac{1}{5}a^2)/\langle (z-z^2)f(z)\rangle.$$

Once again, the minimum eigenvalue is obtained with $a = 0$, and

$$R_c = 2/\langle (z-z^2)f(z)\rangle. \tag{4.5}$$

For the linear profile, $f(z) = 1$ and $R_c = 12$, the known exact value.

For the piecewise-linear profile given by (2.6),

$$R_c = 12/(3\epsilon-2\epsilon^2),$$

which has a minimum value of $12 \times \frac{8}{9} = 10\frac{2}{3}$, attained at $\epsilon = \frac{3}{4}$. For the step-function profile with $f(z) = \delta(z-\epsilon)$,

$$R_c = 2/(\epsilon - \epsilon^2),$$

which has a minimum value of $12 \times \frac{2}{3} = 8$, at $\epsilon = \frac{1}{2}$.

On the other hand, if the upper boundary is at constant pressure rather than being impermeable, while the lower is still impermeable, the boundary conditions are

$$W = DT = 0 \quad \text{at} \quad z = 0, \quad DW = DT = 0 \quad \text{at} \quad z = 1.$$

These are satisfied by $W_1 = 2z - z^2$ and $T_1 = 1$. Equation (4.4) still holds, and gives

$$R = (2 + \tfrac{2}{5}a^2)/\langle(2z - z^2)f(z)\rangle,$$

so that

$$R_c = 2/\langle(2z - z^2)f(z)\rangle.$$

For $f(z) = 1$, we have $R_c = 3$, the known exact result for the linear profile.

For the piecewise-linear profile given by (2.6) we have

$$R_c = 6/(3\epsilon - \epsilon^2).$$

This decreases monotonically as ϵ increases, attaining its minimum value of 3 when $\epsilon = 1$, so that for no value of ϵ does this piecewise-linear profile give a value for R_c less than that for the linear profile.

For the step-function profile with $f(z) = \delta(z - \epsilon)$, we have

$$R_c = 2/(2\epsilon - \epsilon^2),$$

which has a minimum value of 2, at $\epsilon = 1$. Since the constant-pressure condition arises when the porous medium is overlain by a reservoir of fluid, so that the upper boundary condition is not a very restrictive one, these results also are as we would expect.

5. General results and discussion

The single-term Galerkin procedure provides a quick method for obtaining the above results, but in order to demonstrate that they are exact in the limit as the wavenumber a tends to zero, we show how they can be obtained using formal expansions in powers of the small parameter a. We do this for the case of a viscous fluid confined between rigid boundaries; the procedure for the other cases is similar.

We write

$$\left.\begin{aligned} W &= W_0 + aW_1 + a^2W_2 + \ldots, \\ T &= T_0 + aT_1 + a^2T_2 + \ldots, \\ R^{\frac{1}{2}} &= \mathscr{R}_0 + a\mathscr{R}_1 + a^2\mathscr{R}_2 + \ldots \end{aligned}\right\} \tag{5.1}$$

(where the W_i and T_i are now used in a sense different from that in the preceding sections), and substitute in the time-independent form of (2.1), (2.2) and (2.4). The zero-order system of equations,

$$D^4W_0 = 0, \quad D^2T_0 = 0, \tag{5.2a, b}$$

$$W_0 = DW_0 = DT_0 = 0 \quad \text{at} \quad z = 0, 1, \tag{5.2c}$$

has the solution

$$W_0 = 0, \quad T_0 = \text{constant}. \tag{5.3}$$

The order-a system,

$$D^4 W_1 = \mathscr{R}_0 T_0, \quad D^2 T_1 = -\mathscr{R}_0 f W_0,$$
$$W_1 = DW_1 = DT_1 = 0 \quad \text{at} \quad z = 0, 1,$$

yields the solution (with the arbitrary factor suitably chosen)

$$W_1 = z^2 - 2z^3 + z^4, \quad T_1 = \text{constant} \tag{5.4}$$

if

$$\mathscr{R}_0 T_0 = 24. \tag{5.5}$$

The requirement that T_1 be orthogonal to T_0, in the sense that $\langle T_0 T_1 \rangle = 0$, implies that $T_1 = 0$. The order-a^2 system,

$$D^4 W_2 - 2D^2 W_0 = \mathscr{R}_0 T_1 + \mathscr{R}_1 T_0, \tag{5.6a}$$
$$D^2 T_2 - T_0 = -\mathscr{R}_0 f W_1 - \mathscr{R}_1 f W_0, \tag{5.6b}$$
$$W_2 = DW_2 = DT_2 = 0 \quad \text{at} \quad z = 0, 1, \tag{5.6c}$$

together with the requirement that $\langle W_1 W_2 \rangle = 0$, yields

$$W_2 = 0, \quad \mathscr{R}_1 = 0.$$

Further, when we integrate (5.6b) from $z = 0$ to $z = 1$, and use the boundary conditions on T_2, we obtain

$$\langle T_0 \rangle = \mathscr{R}_0 \langle f W_1 \rangle.$$

Using the results (5.4) and (5.5) we then have

$$R \doteqdot \mathscr{R}_0^2 = 24/\langle (z^2 - 2z^3 + z^4) f(z) \rangle.$$

This is the same expression as in (2.5). We now see that the Galerkin method yielded precise values because the trial functions used were, to the lowest order in a, exact.

The expansion procedure can be continued to find $\mathscr{R}_2, \mathscr{R}_3, \ldots$. Since $\mathscr{R}_1 = 0$, the minimum value of $R^{\frac{1}{2}}$ is attained at $a = 0$ if $\mathscr{R}_2 > 0$. The class of functions $f(z)$ for which $\mathscr{R}_2 > 0$ includes $f(z) = 1$ and $f(z) = \delta(z - \frac{1}{2})$, as we find when we perform the necessary calculations. For the linear profile this merely confirms what we already knew, but now we conclude that the value 384 which we obtained in § 2.1 is indeed the critical Rayleigh number for the step-function profile with rigid constant-flux boundaries.

For other functions $f(z)$, $\mathscr{R}_2$ will not be positive, and $R^{\frac{1}{2}}$ will attain its minimum at some non-zero value of a. However, since instability will occur if the Rayleigh number exceeds $\mathscr{R}_0^2$ for at least some disturbances (namely those with very small wavenumber), the value given by our approximation will be an upper bound on the critical Rayleigh number for the onset of instability when disturbances of all wavenumbers are considered. In many cases this upper bound will be close to the precise value. The available evidence indicates that the approximation is good for the piecewise-linear profiles discussed above provided that the discontinuity in gradient does not occur close to a boundary. The degree of accuracy is likely to be poor when $f(z)$ becomes negative somewhere in the range [0, 1], because we expect from physical considerations that W then could have a node within that range, so that convection cells could occur in more than one layer, as in some situations involving 'penetrative convection', and it may then be more appropriate to use a Rayleigh number based on the maximum temperature difference rather than that defined in this paper. However, we believe that when $f(z) \geqslant 0$ for all z in $[0, 1]$ our approximate results are likely to be qualitatively useful.

For such $f(z)$, we now return to the question, raised in the introduction, as to which particular $f(z)$ gives the minimum value of R_c (or M_c). We notice that, in each of the cases considered, $f(z)$ appeared in the term $\langle Wf(z)\,T\rangle$ in the denominator of the expression for R. Also, we were able for small wavenumbers to use the same trial functions for all $f(z)$. Since $\langle Wf(z)\,T\rangle \leqslant (WT)_{\max}$ because $\langle f(z)\rangle = 1$, and since the upper bound is attained with $f(z) = \delta(z-z_m)$, where z_m is the value of z at which WT is a maximum, we conclude that the most unstable basic temperature profile (for which $f(z) \geqslant 0$ everywhere) is the step-function profile for which the step occurs at the level at which W is a maximum (since T is constant in our approximation). At this level the horizontal component of the velocity, which is proportional to DW, is zero, i.e. the velocity is vertical. Thus, for example, for the case of two rigid boundaries through which the heat flux is constant, we deduce that the value of 384 which we found is a lower bound on the critical Rayleigh number, with respect to disturbances of infinite wavelength, for *any* basic temperature profile with temperature gradient non-negative everywhere. Hence the condition $R < 384$ is a sufficient condition for stability, with respect to disturbances of infinite wavelength, for all such temperature profiles. Similar conclusions hold for the other cases considered.

Sometimes, as when the layer is heated from below at a constant rate, we know that the function $f(z)$ is not only non-negative but also decreases (or, for the case of cooling from above, increases) monotonically. We are thus interested in knowing which temperature profile gives the least R_c subject to $f(z) \geqslant 0$, $Df(z) \leqslant 0$ (almost everywhere). We claim that the piecewise-linear profile with $f(z)$ given by (2.6), with ϵ suitably chosen, is the appropriate one (for disturbances of zero wavenumber, at least). In order to demonstrate this, we first note the theorem of Weierstrass, which tells us that any continuous function (such as our basic temperature) can be approximated arbitrarily closely by a piecewise-linear function (with multiple segments). Hence we can approximate the temperature gradient function by a multiple-step function

$$g(z) = g_i \quad \text{when} \quad z_i \leqslant z \leqslant z_{i+1} \quad \text{for} \quad i = 0, 1, \ldots, n, \tag{5.7}$$

where $z_0 = 0$, $z_{n+1} = 1$ and the g_i are constants. Denoting WT by $h(z)$ and letting

$$\int_0^z h(z')\,dz' = H(z),$$

we have, on splitting the range of integration into subintervals (z_i, z_{i+1}) and integrating by parts over each subinterval,

$$\langle Wf(z)\,T\rangle \doteqdot \langle h(z)\,g(z)\rangle = \sum_{i=1}^{n} g_i[H(z_{i+1}) - H(z_i)]. \tag{5.8}$$

Also, since $g(z)$ must satisfy the condition $\langle g(z)\rangle = 1$, we have

$$\sum_{i=1}^{n} g_i(z_{i+1}-z_i) = 1. \tag{5.9}$$

For given z_i and $h(z)$ (and hence given $H(z_i)$) the problem of maximizing the expression (5.8) subject to the constraint (5.9) is a linear-programming problem. A standard theorem (the basis of the simplex algorithm for the solution of such problems) states that the maximum will be obtained when all but one (since there is one constraint) of the g_i are zero. Since we are supposing that the g_i are monotonically decreasing, the non-zero g_i must be g_0. Identifying z_1 with the ϵ in (2.6), we have the required result.

Thus we conclude for the surface-tension problem discussed in § 3, for example, that the value of 34·79 which we obtained is a lower bound for the critical Marangoni number, with respect to infinite wavelength disturbances, for any basic temperature profile with a non-negative, non-increasing gradient. Further, we expect, on the basis of the data of Vidal & Acrivos (1968), that this value will be close to being a lower bound when disturbances of arbitrary wavelength are considered.

In order to be able to say to what extent the results of this paper are typical of problems where the temperature boundary conditions are other than the constant-flux ones, further calculations involving extensive computation appear to be required. Experimental work to confirm the present results is needed. We suggest that using a dissolved substance (such as sugar), rather than heat, as the diffusing quantity should be convenient, especially as the condition of constant (mass) flux could then be satisfied without effort. It is encouraging that the smallest critical Rayleigh number (1340) calculated by Currie (1967), which, by our argument, should be a lower bound on the critical Rayleigh number for heating (at a constant rate) from below with rigid boundaries, is in fact slightly less than the smallest value (about 1400) at which de Graaff & van der Held (1953) obtained convection.

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